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CONVECTIVE FIN OF MINIMUM MASS

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Abstract – The problem of reducing to a minimum the mass of a convective annular/circular fin at the prescribed magnitude of the heat flux is considered. The limiting case of minimization is obtained at the temperature of the outer edge of the fin differing from the temperature of the environment surrounding. The mass of the ideal fin examined differs but little from the mass of the fin of a triangular profile.

NOMENCLATURE

- C, constant of integration;
- D, diameter;
- Q, heat flux per 1 m of fin base [W/m];
- V, fin volume per 1 m of fin base [m^2];
- l, fin height (see Fig. 1);
- q, $= Q/Q_1$, dimensionless heat flux;
- r, radius (see Fig. 1);
- v, dimensionless volume, see equation (6);
- x, dimensionless coordinate, see equation (16);
- α, heat-transfer coefficient;
- Δ, $= \delta/\delta_1$, dimensionless fin thickness;
- δ, fin thickness (see Fig. 1);
- η, fin efficiency;
- ϑ, excessive fin temperature relative to the temperature of the surrounding medium;
- θ, $= \vartheta/\vartheta_1$, dimensionless temperature;
- λ, thermal conductivity;
- ρ, $= r/l$, dimensionless radius;
- σ, fin parameter, see equation (2);
- φ, $= D_2/D_1$, diameter ratio;
- χ, fin parameter, see equation (4).

Subscripts
- 1, fin base;
- 2, outer edge of fin.

INTRODUCTION

A GREAT variety of fins is possible which would differ in size and shape but afford the same heat dissipation. The problem is frequently optimized by the condition that a minimum of material (mass) be expended for fabrication of a fin with the prescribed heat dissipation. For fins made of uniform material this problem is reduced to the condition of a minimum volume. The fins of a specified geometry are optimized on the principle of determining their minimum volume. This offers the possibility for a fin having the same heat dissipation, but a different shape of the cross section, to be of even smaller volume. Therefore, the problem of complete minimization of the volume reduces to that of finding the fin shape which is realized by the methods of the calculus of variations.

For a straight fin the latter problem was solved by Schmidt [1] as long ago as 1926 using the principle of a constant heat flux. The results of this work were confirmed by Duffin [2, 3] who applied a rigorous variational approach. According to these publications, the fin profile is defined by the equation of a parabola and has a zero thickness at the outer edge with a temperature not differing from that of the environment surrounding ($\vartheta_2 = 0$). The above works also consider the problem of optimization of a circular fin, but the condition $\vartheta_2 = 0$ applied proved to be invalid in this case. Moreover, the problems of a straight and an annular/circular fin are considered as separate ones with the result that comparison between the data on these types of fins in the limiting case of the increasing inside and outside radii becomes complicated. A great number of works have been reported to date in which different aspects of the problem of fin optimization are investigated with a variety of additional conditions [1–7]. However, the above shortcomings are typical of all these works.

In this paper, consideration is given to the problem of optimization of a convective annular/circular fin with regard for the remarks made and with application of a one-dimensional mathematical model which, according to [8], is permissible for sufficiently small values of the Bi number.

MATHEMATICAL STATEMENT OF THE PROBLEM

The dependence of the dimensionless temperature, θ, dimensionless heat flux, q, and dimensionless fin thickness, Δ, on the radius, ρ, is described by the equations and additional conditions which incorporate the following dimensionless parameters of the fin:

$$\varphi = \frac{r_2}{r_1}, \tag{1}$$

$$\sigma = \frac{\alpha l^2}{\lambda \delta_1}, \tag{2}$$

$$\eta = \frac{Q_1}{(\varphi + 1)\alpha \vartheta_1 l}. \tag{3}$$

The fin efficiency η is defined as the ratio of the mean fin temperature $\bar{\vartheta}$ to the temperature of its base $\eta = \bar{\vartheta}/\vartheta_1$. It is frequently used to calculate heat dissipation of the fin by Newton's formula. For the purpose of performing an analytical investigation, the following parameter can be used:

$$\chi = \frac{Q_1 l}{\lambda \vartheta_1 \delta_1}, \tag{4}$$

which is related to the previous parameters by the formula

$$\chi = (\varphi + 1)\sigma\eta. \tag{5}$$

The fin volume is expressed in dimensionless form as

$$v = \frac{V}{V_0} \tag{6}$$

where

$$V_0 = \frac{Q_1^3}{2\lambda\alpha^2\vartheta_1^3}. \tag{7}$$

The equation for the radial thermal conductivity of the fin (Fig. 1) is expressed, with equations (1)–(7) taken into account, in the form

$$\chi q = -(\varphi - 1)\Delta\rho\frac{d\theta}{d\rho}, \tag{8}$$

while the equation for heat transfer from the fin surfaces is written as

$$\chi\frac{dq}{d\rho} = -2(\varphi - 1)\sigma\rho\theta. \tag{9}$$

The variables q, θ, and Δ in equations (8) and (9) are the functions of the independent variable ρ. Their values for the inner boundary of an annular/circular fin at $\rho = \rho_1 = 1/(\varphi - 1)$ are obtained directly from definitions of the dimensionless variables

$$q_1 = \theta_1 = \Delta_1 = 1. \tag{10}$$

Moreover, it follows from (8) and (9) that

$$\left(\frac{d\theta}{d\rho}\right)_{\rho=\rho_1} = -\chi, \tag{11}$$

$$\left(\frac{dq}{d\rho}\right)_{\rho=\rho_1} = -\frac{2\sigma}{\chi}. \tag{12}$$

An additional condition for the outer edge of the fin at $\rho = \rho_2 = \varphi/(\varphi - 1)$ is the absence of heat flux,

$$q = 0. \tag{13}$$

In order to find a particular solution for the system of equations (8)–(9), it is necessary to have, besides (13), a second additional condition for θ at the outer fin edge at $\rho = \rho_2$. Since ordinarily the thickness of the optimized fins $\Delta_2 = 0$, it follows from (8) and (9) that θ_2 at the outer fin edge should not necessarily be $\theta_2 = 0$. Therefore, the second condition to determine the integration constants in (8) and (9) can be expressed at $\rho = \rho_2$ in a general form as

$$\theta = \theta_2. \tag{14}$$

And finally, in order that the system of equations (8)–(9) could be integrated, the function $\Delta = \Delta(\rho)$ should be known.

The problem of optimization was considered in detail in [1, 2]. By solving the variational problem it was proved that the profile of a fin of minimum volume is characterized by a constant temperature gradient. This leads to a linear temperature distribution over the radius

$$\theta = C' + C''\rho. \tag{15}$$

Replacing the coordinate ρ by

$$x = \frac{\varphi}{(\varphi - 1)} - \rho \tag{16}$$

and applying the conditions (10), (11) yield

$$\frac{d\theta}{dx} = \chi = \text{constant}, \tag{17}$$

$$\theta = (1 - \chi) + \chi x. \tag{18}$$

Thus, for the outer fin edge we get from (14) and (18)

$$\theta_2 = 1 - \chi. \tag{19}$$

It follows from (19) that if the temperature is distributed according to (18), the quantity θ_2 is unambiguously determined in terms of χ and is a dimensionless parameter of the fin, like the foregoing parameters χ, σ, η, and φ.

Equations (17) and (18) are expressed in terms of θ_2 as

$$\frac{d\theta}{dx} = 1 - \theta_2, \tag{20}$$

$$\theta = \theta_2 + (1 - \theta_2)x. \tag{21}$$

Equations (9) and (13) give for the heat flux:

$$q = \frac{2\sigma}{1 - \theta_2}\left\{\varphi\theta_2 x + [\varphi(1-\theta_2) - (\varphi-1)\theta_2]\frac{x^2}{2} - (\varphi-1)(1-\theta_2)\frac{x^3}{3}\right\}. \tag{22}$$

The condition $q_1 = 1$, (10), interrelates the fin parameters as

$$1 = \frac{2\sigma}{1 - \theta_2}\,\frac{(\varphi + 2) + (2\varphi + 1)\theta_2}{6} \tag{23}$$

and allows elimination of σ from (22):

$$q = \frac{6}{(\varphi+2) + (2\varphi+1)\theta_2}\left\{\varphi\theta_2 x + [\varphi(1-\theta_2) - (\varphi-1)\theta_2]\frac{x^2}{2} - (\varphi-1)(1-\theta_2)\frac{x^3}{3}\right\}. \tag{24}$$

According to (8), the fin profile is determined thus:

$$\Delta = \frac{q}{\varphi - (\varphi - 1)x}. \tag{25}$$

It follows from (25) that, by virtue of (13), the fin thickness at the outer edge should really be $\Delta_2 = 0$.

In the above equations, the solutions depend on the dimensionless fin parameters, which amount to five in the present paper, viz. φ, χ, σ, η, θ_2. These parameters are interrelated by equations (5), (19), and (23). Thus, an unambiguous solution of the problem is obtained by assigning the values of two parameters. The problem as stated, when for an annular/circular fin with the known φ it is necessary to find the condition of the minimum volume, requires still another (fourth) equation relating the fin parameters. This is the condition of the fin volume minimization

$$v = \frac{2(\varphi - 1)\sigma^2}{\chi^3}\int_{\rho_2}^{\rho_2} \Delta\rho \, d\rho = \text{minimum}. \quad (26)$$

ANNULAR/CIRCULAR FIN OPTIMIZATION

In view of (8), (16) and (17), equation (26) can be expressed as

$$v = \frac{2\sigma^2}{\chi^3}\int_0^1 q \, dx = \min, \quad (27)$$

where the integral is easily integrated, with (24) taken into account, and has the form

$$\int_0^1 q \, dx = \frac{2\sigma}{\chi} \frac{(\varphi + 1) + (3\varphi + 1)\theta_2}{12}. \quad (28)$$

Hence for the volume from (19), (23), (27), and (28) we have

$$v = \frac{9[(\varphi + 1) + (3\varphi + 1)\theta_2}{(1 - \theta_2)[(\varphi + 2) + (2\varphi + 1)\theta_2]^3}, \quad (29)$$

while the minimization of v with respect to θ_2 yields

$$\frac{\partial v}{\partial \theta_2} = 0. \quad (30)$$

Realization of (30) provides an additional equation to determine θ_2:

$$3(3\varphi + 1)\theta_2^2 - 2(\varphi - 1)\theta_2 - (\varphi - 1) = 0. \quad (31)$$

The system of equations (5), (19), (23), and (31) gives all of the optimized fin parameters, i.e. φ, σ, χ, η, θ_2, provided one of these is known.

After the solution of this system of equations, the assumption that $\theta_2 = 0$ at the outer edge of the fin yields: $\varphi = 1, \chi = 1, \sigma = 1, \eta = 1/2$. Since the limiting case, $\varphi = 1$, corresponds to a fin over a straight base, then the condition that $\theta_2 = 0$ is not true in the case of the most minimum volume. It is interesting to note that it is at this point that a lot of misunderstandings and erroneous view-points are encountered even in the contemporary scientific literature.

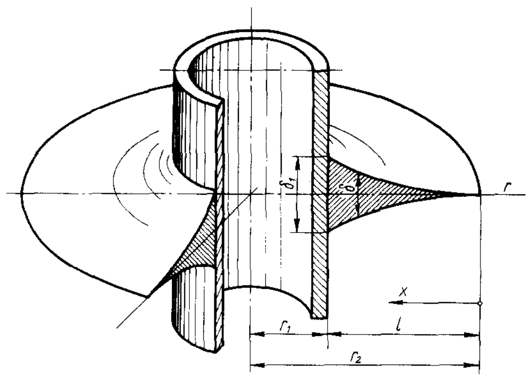

FIG. 1. Schematic diagram of an annular/circular fin ($\varphi = 3$).

This problem was investigated for the first time by Schmidt [1]. Prior to the problem of an annular/circular fin, Schmidt considered in his works the problem of optimization of the fin having a straight base. In this case the temperature is first expressed in the general form as a linear function. The solution of the differential heat conduction equation gives the formula for the fin thickness and then an equation is composed for the fin volume {equation (18) p. 888 of [1]} containing the unknown constants. For these to be determined, Schmidt uses the formula for the heat flux and resorts to minimization of his formula (18). Only then the conclusion is drawn that the temperature difference at the outer edge of the fin with a straight base should equal to zero for the minimum volume to be realized. As regards the works of Schmidt or annular/circular fins, these are less consistent. For some unknown reasons, Schmidt used the result obtained for the fin with a straight base and automatically and without special verification adapted the condition $\theta_2 = 0$ to an annular/circular fin. The formulae obtained by Schmidt in this way are cited in the contemporary literature and are recommended as formulae for an isogradient annular/circular fin of the most minimum volume at the prescribed heat dissipation. As it is, the assumption that $\theta_2 = 0$ is

Table 1. Comparison between the dimensionless volumes of annular/circular fins with minimum mass determined from Schmidt's formulae [1] and the formulae of the present author [(29], (31)] and circular fins of triangular profile [10]

φ	Dimensionless volume v Schmidt's data	Present author's data	Triangular cross section
1	0.6667	0.6667	0.7037
2	0.4219	0.3937	0.4143
3	0.2880	0.2510	0.2708
4	0.2083	0.1725	0.1911
5	0.1574	0.1255	0.1426
6	0.1230	0.0952	0.1109

Table 2. Optimum parameters of annular/circular fins

	Profile according to equation (25)			Triangular profile			Fin of constant thickness		
φ	σ	η	v	σ	η	v	σ	η	v
1	1.0000	0.5000	0.6667	0.8574	0.5917	0.7037	1.0071	0.6267	1.0085
1.5	0.5497	0.5771	0.5094	0.6143	0.6116	0.5312	0.7657	0.6362	0.8115
2	0.4084	0.5950	0.3937	0.4941	0.6225	0.4143	0.6328	0.6426	0.6618
2.5	0.3267	0.6048	0.3111	0.4232	0.6296	0.3312	0.5507	0.6462	0.5493
3	0.2727	0.6111	0.2510	0.3758	0.6349	0.2708	0.4944	0.6488	0.4628
4	0.2053	0.6188	0.1725	0.3169	0.6416	0.1911	0.4208	0.6522	0.3426
5	0.1647	0.6234	0.1255	0.2814	0.6457	0.1426	0.3741	0.6546	0.2646
6	0.1376	0.6265	0.0952	0.2571	0.6484	0.1109	0.3410	0.6564	0.2116
∞	0	$\frac{11+2\sqrt{10}}{27}$	0	0	2/3	0	0	2/3	0

Note: $\varphi = 1$ corresponds to a fin with a straight base.

equivalent to replacing equation (31) by an arbitrary additional condition to determine the dimensionless fin parameters. The fact that $\theta_2 = 0$ is in error is vividly illustrated in Table 1, where the dimensionless volumes of fins are presented and comparison is made between the case considered in the present paper, that described by Schmidt ($\theta_2 = 0$) and the case of the fin of a triangular cross section (non-isogradient profile).

It is seen from Table 1 that at $\varphi = 1$ the results of Schmidt and the results of the present author coincide. But already at $\varphi = 2$ Schmidt's formulae overestimate the values of v not only as compared with the results of the present paper, but also with the volume of the fin of a triangular cross section.

CONCLUSION

As a result of the foregoing analysis, the method has been developed for determining the optimum fin with a minimum volume at the prescribed heat flux. Since fabrication of complex-profiled fins (Fig. 1) involves some difficulties, the fins of constant thickness have found application in practice as well as the fins of triangular and trapezoidal profiles. In this regard, the optimized fin investigated above may be looked upon as an idealized standard for the determination of the adequacy of fins of other shapes. Table 2 contains the parameters σ, η, and relative volumes v. The parameters are presented for a convective annular/circular fin with a profile described by (25) and for annular/circular fins with a minimum volume having a constant thickness or a triangular profile [9, 10]. It follows from Table 2 that the parameters of the fin with a triangular profile are only slightly worse than those of an ideal fin. The volume of a fin with a triangular profile exceeds that of an ideal one by about 5.6% at $\varphi = 1$ and by 16.5% at $\varphi = 6$.

The data in Table 2 can be used in designing finned heating surfaces since the principal dimensions of the optimum fins at the prescribed φ are determined by formulae (2) and (3) in the present paper,

$$l = \frac{q_1}{\alpha \vartheta_1 \eta(\varphi + 1)}, \tag{32}$$

$$\delta_1 = \frac{\alpha l^2}{\lambda \sigma}, \tag{33}$$

$$r_1 = \frac{l}{(\varphi - 1)}. \tag{34}$$

When the radius r_1 of a cylinder (tube) with an annular/circular fin is assigned, then formulae (32) and (34) yield

$$(\varphi^2 - 1)\eta = \frac{Q_1}{\alpha \vartheta_1 r_1}. \tag{35}$$

On having supplemented Table 2 with the column $(\varphi^2 - 1)\eta$ we may interpolate the optimum values of φ, σ, and η, and then determine l and δ_1 from formulae (33) and (34).

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CONVECTION D'UNE AILETTE DE MASSE MINIMALE

Résumé — On considère le problème de la réduction de la masse d'une ailette annulaire/circulaire et convectante pour une valeur donnée du flux thermique. Le cas limite de l'optimum est obtenu pour une température du bord extérieur différente de celle de l'environnement. La masse de l'ailette idéale diffère légèrement de la masse d'une ailette à profil triangulaire.

EINE WÄRMEÜBERTRAGUNGSRIPPE MIT MINIMALER MASSE

Zusammenfassung—Es wird das Problem behandelt, die Masse einer kreisringförmigen Wärmeübertragungsrippe bei vorgegebenem Wärmestrom auf ein Minimum zu reduzieren. Für das ermittelte Minimum unterscheidet sich die Temperatur an der Rippenspitze von der Umgebungstemperatur. Die Masse der ermittelten idealen Rippe weicht jedoch nur wenig von derjenigen einer Rippe mit dreieckigem Querschnitt ab.

КОНВЕКТИВНОЕ РЕБРО С МИНИМАЛЬНОЙ МАССОЙ

Аннотация — Рассматривается проблема минимизации массы конвективного круглого ребра при заданной величине теплового потока. Предельный случай минимизации получен при температуре внешней границы ребра, не равной температуре окружения: Масса рассмотренного идеального ребра отличается лишь незначительно от массы ребра треугольного профиля.